Field Embeddings which are conjugate under a unit of a p-adic classical Group

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September 11, 2012

Abstract

Let (V, h) be a signed Hermitian space over a division algebra D which is of index at most two over a non-Archimedean local field k of residue characteristic not 2. Let G be the unitary group defined by h and σ the adjoint involution. Suppose we are given two σ-invariant field extensions E_1 and E_2 of k in $\mathrm{End}_D(V)$ which are isomorphic as k-algebras with involutions and suppose that there is a point x in the Tits building $\mathfrak{B}(G)$ which is fixed by $E_1^\times$ and $E_2^\times$. If we fix a certain map from E_1 to k, then the pair (E_i, σ) defines a unique extension of h to a signed Hermitian $E_1 \otimes_k D$-form $\tilde{h}_i$ on V. Then any k-algebra isomorphism from (E_1, σ) to (E_2, σ) can be realized through conjugation by an element of the stabilizer of x in G if both (E_1, x) and (E_2, x) have the same Broussous-Grabitz embedding type and $\tilde{h}_1$ is isomorphic to $\tilde{h}_2$.

1 Introduction

This article is about a Skolem-Noether kind of lemma in the frame of p-adic classical groups in the case of odd residue characteristic. For general linear groups this kind of lemmas encoded the invariants for analyzing the rigidity of certain irreducible representations on open compact subgroups of a p-adic group. These representations, called simple types, are used to construct and classify supercuspidal representations of the group of interest. The rigidity question for two simple types is how they are related if they are contained the same supercuspidal representation. For example I want to mention the work of Broussous and Grabitz [BG00] which is used in the above classification question for $\mathrm{GL}_m(D)$. For completeness let us mention that the latter is mainly done by Secherre, Stevens and Broussous in the frame of Bushnell-Kutzko theory. Stevens constructed all supercuspidal representations for p-adic classical groups in the non-quaternion algebra case [Ste08]. To understand were the Skolem-Noether like proposition is involved let us give more details. Let H be a p-adic group of the kind mentioned above. A simple type itself is constructed by a combinatorial algebraic object, called simple stratum, which especially consists of a hereditary order $\mathfrak{a}$ in an Azumaya algebra A over a non-Archimedean local field k and a field extension $E|k$ in A such that $E^\times$ normalizes $\mathfrak{a}$. Note that A always is the endomorphism ring of the vector space on which H is defined. We call such a pair $(E, \mathfrak{a})$ an embedding. An approach to the rigidity question for simple types is to study a certain rigidity of simple strata, which leads us to the classification of classes of embeddings with same hereditary order under conjugation by elements of $\mathfrak{a}^\times \cap H$. These classes are in one to one corresponding to certain numerical invariant which we call embedding type. For example in [BG00] these embedding types are given for the case if H is $A^\times$, i.e. some $\mathrm{GL}_m(D)$. From their article it is easy to deuce the following proposition.

Proposition 1.1 *If two embeddings $(E_i, \mathfrak{a})$ of A with k-algebra isomorphic fields and the same hereditary order $\mathfrak{a}$ have the same embedding type then any k-algebra isomorphism between the two fields can be realized by a conjugation with an element of $\mathfrak{a}^\times$.*

This article provides the analog proposition for p-adic groups. Here we consider a p-adic unitary group G defined by a signed Hermitian space (V, h) over a central division algebra of finite index over a non-Archimedean local field k of odd residue characteristic. We consider embeddings which are invariant under the adjoint involution σ of h. Here A is $\mathrm{End}_D(V)$. A non-trivial σ-equivariant k-linear map λ_E from E to k for some $E|k$ in A defines a Hermitian form $\tilde{h}$ om the $E \otimes_k D$-module V in the manner of [BS09]. For simplicity one assumes an extra condition on λ_E. The desired proposition which we prove here is:

Proposition 1.2 *If two invariant embeddings $(E_i, \mathfrak{a})$ of A with k-algebra σ-equivariantly isomorphic fields and the same hereditary order $\mathfrak{a}$ have the same embedding type and isomorphic Hermitian spaces $(V, \tilde{h}_i)$ then any k-algebra σ-equivariant isomorphism between the two fields can be realized by a conjugation with an element of $\mathfrak{a}^\times \cap G$.*

I thank Prof. S. Stevens for giving me the hint to this topic and the DFG for financing my position at University Muenster for that project.

Notation 1.3 Let D be a skew field of index one or two whose center is a non-Archimedean local field k of odd residue characteristic, and let ρ be an involution on D. We fix a non-degenerate ϵ-Hermitian form h on an m-dimensional Right D-vector space V with adjoint involution σ. We denote the fixed field of k under σ by k_0 and $\mathrm{U}(h)$ by G. The dual of ? with respect to h is written as $?^{\#,h}$, and if there is no confusion about h we write $?^{\#}$ (see [BS09]). We write $\tilde{G}$ for $\mathrm{GL}_D(V)$.

We are given two k-algebra monomorphisms

$$\phi_i : \ (E, \sigma') \hookrightarrow (\mathrm{End}_D(V), \sigma), \ i = 1, 2$$

which are σ'-σ-equivariant. Let E_0 be the set of σ'-fixed points. Let Λ be an element of $\mathfrak{B}(G)$, i.e. a self-dual lattice function.

Definition 1.4 Let $\mathfrak{a}_\Lambda$ be the square lattice function of Λ and $\mathfrak{g}_\Lambda$ be the Moy-Prasad filtration.

1. $\mathfrak{N}(\Lambda) := \{g \in \mathrm{GL}_D(V) \mid \ g\mathfrak{a}_\Lambda g^{-1} = \mathfrak{a}_\Lambda\}$.
2. $\mathrm{U}(\Lambda) := \mathfrak{N}(\Lambda) \cap G = \{g \in G \mid \ g\mathfrak{g}_\Lambda g^{-1} = \mathfrak{g}_\Lambda\}$.

Question: When does there exists an element g of $\mathrm{U}(\Lambda)$, such that ϕ_1 is conjugated to ϕ_2 via g?

In the manner of Broussous and Stevens given in [BS09] we fix a non-zero k-linear σ'-σ-equivariant map λ from E to k such that

$$\mathfrak{p}_{E_0} = \{e \in E_0 \mid \ \lambda(eo_{E_0}) \subseteq \mathfrak{p}_{k_0}\}. \tag{1}$$

We attach to ϕ_i an ϵ-hermitian form

$$\tilde{h}_i := \tilde{h}_{\phi_i} : V \times V \to E \otimes_k D$$

with respect to $\sigma' \otimes_k \rho$ such that

$$h = \overline{\lambda \otimes_k \mathrm{id}_D} \circ \tilde{h}_i. \tag{2}$$

For the proof see [BS09]. We recall, for two linear forms f_1 and f_2,

$$\overline{f_1 \otimes_k f_2}(v \otimes_k w) := f_1(v) f_2(w).$$

Proposition 1.5 *Suppose that $\phi_i(E^\times) \subseteq \mathfrak{N}(\Lambda)$, for $i \in \{1, 2\}$, and $(V, \tilde{h}_1)$ is isomorphic to $(V, \tilde{h}_2)$ as hermitian $E \otimes_k D$ spaces, suppose further that the pairs $(\phi_i(E), \Lambda)$ have the same Broussous-Grabitz embedding types. Then there is an element g of $\mathrm{U}(\Lambda)$ such that*

$$\phi_1^g(x) := g\phi_1(x)g^{-1} = \phi_2(x),$$

for all $x \in E$.

Remark 1.6 Having the same Broussous-Grabitz embedding types [BG00] is equivalent in saying that there is an element u of $\mathfrak{a}$ such that

$$\phi_1^u(x) := \phi_2(x).$$

The difference is that a priori that element does not lie in G.

We follow the proof given in [BH96, 1.6]. Given a lattice chain L there the sequence of natural numbers

$$d_i(L) := \dim_{\kappa_D}(L_i/L_{i+1})$$

is called the *invariant* of $\mathcal{L}$. Analogously we define d_i for lattice sequences and lattice functions. We call a lattice chain L on V self-dual with respect to h if L equals either $(L^{\#}_{-i})_{i\in\mathbb{Z}}$ (type I) $(L^{\#}_{-i+1})_{i\in\mathbb{Z}}$ (type II) or

(type II).

In particular we have

$$L_0^{\#} = L_0 \text{ (I) or } L_1^{\#} = L_0 \text{ (II)}.$$

Lemma 1.7 If two self-dual lattice chains L and L' on V have the same type and the same invariants, then there is an element g of G such that the lattice chains gL and L' equal.

Proof: There is a Witt decomposition $\{W_i \mid \ i \in I\}$ of V with respect to h which splits both lattice chains. W.l.o.g. we can assume that the anisotropic part W_0 of the Witt decomposition is trivial, because L and L' equal on W_0 by [BT87, 2.9]. Let r be the period of L. We choose a decomposition of I into two disjoint set I^+ and I^- such that

$$\sigma(I^+) = I^-.$$

Further we define

$$W^+ := \oplus_{i\in I^+} W_i,\ W^- := \oplus_{i\in I^-} W_i$$

and

$$L^+ := L \cap W^+,\ L^- := L \cap W^-.$$

Let $\mu(L, j)$ be the set of indexes $i \in I$ for which $W_i \cap L_j$ differs from $W_i \cap L_{j+1}$. Analogously we define $\mu(L^+, j)$, but this set can be empty.

Case 1: We assume that L is of type (I) and r is even. We choose, for $0 \le j < \frac{r}{2}$, injective maps

$$\phi_j^+ :\ \mu(L^+, j) \to \mu(L', j),\ \phi_j^- :\ \mu(L^-, j) \to \mu(L', j) \setminus \operatorname{im}(\phi_j^+).$$

Such a choice is possible because $d_j(L)$ equals $d_j(L')$. We define

$$I'^+ := \bigcup_{0\le j<\frac{r}{2}} (\operatorname{im}(\phi_j^+) \cup \sigma(\operatorname{im}(\phi_j^-))),$$

and we put I'^- to be the complement of I'^+ in I. Because of

$$i \in \mu(L', j) \text{ if and only if } \sigma(i) \in \mu(L', -j-1), \tag{3}$$

for all $i \in I$, we have that $I'^+ \cap \sigma(I'^+)$ is empty, and by symmetry $I'^- \cap \sigma(I'^-)$ is empty too. Thus

$$\sigma(I'^+) = I'^-.$$

This new decomposition of I defines

$$W'^+ := \oplus_{i\in I'^+} W_i,\ W'^- := \oplus_{i\in I'^-} W_i,\ L'^+ := L' \cap W'^+,\ L'^- := L' \cap W'^-.$$

By construction L'^{+} and L^{+} are lattice sequences with the same invariants, and there is an isomorphism u of D-vector spaces from W^{+} to $W^{'+}$ such that uL^{+} equals L'^{+}. The map

$$g := (u,0) + \sigma((u^{-1},0)) : W^{+} \oplus W^{-} \to W^{'+} \oplus W^{'-}$$

is an element of G and gL equals L'.

Case 2: The type of L is (I) and r is odd. We can construct $W^{'+}$ and $W^{'-}$ as in case 1, but the only think we have to change is the definition of $\phi^{+}_{\frac{r-1}{2}}$. Because of (3) the set $\mu(L', \frac{r-1}{2})$ is invariant under the action of σ, i.e. since L and L' have the same invariants we can choose $\phi^{+}_{\frac{r-1}{2}}$ such that

$$\sigma(\mathrm{im}(\phi^{+}_{\frac{r-1}{2}})) \cap \mathrm{im}(\phi^{+}_{\frac{r-1}{2}}) = \emptyset.$$

We now conclude as in case 1.

Case 3: The type of L is (II). Different to the cases before we have

$$i \in \mu(L', j) \text{ if and only if } \sigma(i) \in \mu(L', -j), \tag{4}$$

We follow the proof of the cases 1 and 2, but with the following differences:

1. We consider $0 \leq j \leq \frac{r}{2}$, i.e. if r is even the index $\frac{r}{2}$ is considered too in all formulas.
2. The set $\mu(L', 0)$ is σ-equivariant.
3. If r is even the set $\mu(L', \frac{r}{2})$ is σ-equivariant.

For the σ-equivariant sets we apply the procedure of case 2 for the choice of the map ϕ^{+}_{j}. After these preparations we conclude as in case one to finish the proof. q.e.d.

We now proof the proposition for $D = k$.

Proof: [of 1.5 for $D = k$] We only need to consider the self-dual lattice chain L which satisfies

$$\{L_i \mid \ i \in \mathbb{Z}\} = \mathrm{im}(\Lambda).$$

As in [BH96, 1.6]. we consider V as a hermitian E-vector space V_i via ϕ_i and $\tilde{h}_i$. By [BS09, 5.5] we have

$$L_i^{\#,h} = L_i^{\#,h_1} = L_i^{\#,h_2},$$

for all integers i. Thus L seen as an o_E-lattice chain in V_1 has the same type and the same invariants as it has in V_2. Thus, since V_1 and V_2 are isomorphic hermitian spaces and because of Lemma 1.7, there is an isomorphism u of hermitian spaces from V_1 to V_2 such that

$$u(L_i) = L_i$$

for all integers i. By (2) the map u is an element of G, and being E-linear implies

$$\phi_1^{u} = \phi_2.$$

q.e.d.

Assumption 1.8 *From now on, we assume $D \neq k$. This implies that k_0 equals k by [Sch85, 10.2.2]. Thus we have two cases left.*

1. *$[E : k]$ is odd, i.e. $E \otimes_k D$ is a skew field. In particular is σ' the identity and $E = E_0$.*

2. $[E:k]$ is even, i.e. $E\otimes_k D$ is E-algebra isomorphic to $\mathrm{M}_2(E)$.

Case 1: We can apply the proof of 1.5 for $D=k$, but we have to study $o_{E\otimes_k D}$ to generalize the proof of Lemma [BS09, 5.5] to this case. We write Δ for $E\otimes_k D$. We fix prime elements π_E and π_D of E and D, respectively. Let f and e be the inertia degree and the ramification index of $E|k$, respectively.

Lemma 1.9 We extend the valuation ν in k to a valuation, also denoted by ν, on $o_{E\otimes_k D}$ and it equals to $\nu\otimes_k\nu$ defined via

$$(\nu\otimes_k\nu)(t):=\sup\{\inf\{\nu(e_i)+\nu(d_i)\mid\ i\in I\}\mid\sum_{i\in I}e_i\otimes_k d_i=t\}$$

(see [BT84, (19)]). Further we have:

1. The element
$$\pi_\Delta:=\pi_E^{\frac{1-e}{2}}\otimes_k\pi_D$$
is a prime element of Δ.
2. The valuation ring of Δ is
$$o_E\otimes_{o_k}o_D+\mathfrak{p}_E^{\frac{1-e}{2}}\otimes_{o_k}\mathfrak{p}_D.$$
3. The valuation ideal of Δ is
$$\mathfrak{p}_E\otimes_{o_k}o_D+\mathfrak{p}_E^{\frac{1-e}{2}}\otimes_{o_k}\mathfrak{p}_D.$$

Proof: We choose elements a_1, a_2 of o_D and $b_i,\ 1\le i\le f$ of o_E such that $(\bar a_1,\bar a_2)$ is a κ_k-basis of κ_D and $(\bar b_1,\ldots,\bar b_f)$ a κ_k-basis of κ_E. because of the coprime degrees the tuple of products

$$(\bar a_i\bar b_j)_{i,j}$$

is a κ_k-basis of κ_Δ. The element π_Δ is an a prime element of Δ, because

$$\nu(\pi_\Delta)=\frac{1-e}{2e}+\frac{1}{2}=\frac{1}{2e},$$

and the tuple

$$(\pi_\Delta^l a_i b_j)_{0\le l\le 2e-1,1\le i\le 2,1\le j\le f}$$

is a splitting basis of ν on $D|k$, i.e. we obtain 1. and 2.. By the Chinese reminder theorem the valuations

$$\nu(\pi_E^{l_1}\otimes_k\pi_D^{l_2}),\ l_1\in\{0,\ldots,e-1\},\ l_2\in\{0,1\}$$

form a system of representatives of $\mathbb{Z}/(2e\mathbb{Z})$. And thus by [BT84, (21)] the valuation ν on Δ equals to $\nu\otimes_k\nu$. q.e.d.

Lemma 1.10 Analogous to [BS09, 5.5] Let ϕ be a k-algebra monomorphisms from E to $\mathrm{End}_D(V)$ and assume that $[E:k]$ is odd. Let M be a full o_D-lattice of V and assume M to be an o_E-lattice via ϕ. We then have

$$M^{\#,h}=M^{\#,\tilde h_\phi}.$$

Proof: We define $\tilde\lambda$ to be $\overline{\lambda\otimes_k\mathrm{id}_D}$. We repeat the argument of [BS09, 5.5].

$$\begin{aligned}M^{\#,h}&=\{v\in V\mid\ h(v,M)\subseteq\mathfrak{p}_D\}\\&=\{v\in V\mid\ \tilde\lambda(h(v,M))\subseteq\mathfrak{p}_\Delta\}\\&=M^{\#,\tilde h_\phi}.\end{aligned}$$

To prove the second equality we need to show

$$\mathfrak{p}_\Delta = \{x \in \Delta \mid \ \lambda(xo_\Delta) \subseteq \mathfrak{p}_D\}.$$

$\subseteq$: This inclusion follows from statement 2 of Lemma 1.9, the equality (1) and the identities

$$\pi_\Delta(o_E \otimes_{o_k} o_D) = \mathfrak{p}_E^{\frac{1-e}{2}} \otimes_{o_k} \mathfrak{p}_\Delta \tag{5}$$

and

$$\pi_\Delta(\mathfrak{p}_E^{\frac{1-e}{2}} \otimes_{o_k} \mathfrak{p}_D) = \pi_k \mathfrak{p}_E^{1-e} \otimes_{o_k} o_\Delta = \mathfrak{p}_E \otimes_{o_k} o_\Delta.$$

$\supseteq$: This follows because the o_Δ-module $\mathfrak{J}$ does not contain 1_Δ by (1). q.e.d.

Proof: [of 1.5 for case 1 on $D \neq k$.] Because of Lemma 1.10 the proof for the case $D = k$ generalizes to this case. q.e.d.

Case 2: If Proposition 1.5 is true then the corresponding field embeddings are conjugate under a unit of $\mathfrak{a}$ and this means that the Broussous-Grabitz embedding types [BG00] have to be the same. To prove Proposition 1.5 we firstly generalize Lemma 1.7.

Lemma 1.11 If two self-dual o_D-lattice functions Λ and Λ' on V have the same invariants and are o_E-lattice functions with same Broussous-Grabitz embedding types, then there is an element h of the centralizer H of E in G, such that the lattice functions $g\Lambda$ and Λ' equal.

This case splits into two subcases.

2.1 The inertia degree $f(E|k)$ is odd. Note that here the embedding types are automatically the same.

2.2 The inertia degree $f(E|k)$ is even.

Case 2.1: Let ϕ be an equivariant embedding of (E, σ') into $\mathrm{End}_D(V)$. Here we can find the following situation. There is a two-dimensional unramified and ρ-invariant field extension L in D, and an element p_E of E such that p_E^2, which we denote by π_k, is a prime element of k. Further we find a square root π_D of π_k in D, which normalizes L, such that $\rho(\pi_D)$ is either $+\pi_D$ of $-\pi_D$. We denote the non-trivial element of $\mathrm{Gal}(L|k)$ by τ. Let W an arbitrary $\frac{\dim_E(V)}{2}$-dimensional E-vector space. We define a right-D-action on $W \otimes_k L$ via

$$(w \otimes_k l).\pi_D := (p_E w \otimes_k l).$$

As a $E \otimes_k D$-module the space $W \otimes_k L$ is isomorphic to V by the theory of semi-simple modules. Thus we identity both of them. We now prove lemma 1.11 for Case 2.1. .

Proof: Step 1: The algebra $E \otimes_k D$ is E-algebra isomorphic to $\mathrm{M}_2(E)$ via

$$\pi_D \mapsto \begin{pmatrix} p_E & 0 \\ 0 & -p_E \end{pmatrix}, l' \mapsto \begin{pmatrix} 0 & l'^2 \\ 1 & 0 \end{pmatrix},$$

where l' is a unit in L which satisfies $\tau(l') = -l'$. We identify $E \otimes_k D$ with $\mathrm{M}_2(E)$. Let $\mathfrak{M}$ be the image of $W \times W$ under $\tilde{h}$. For elements A of $\mathfrak{M}$ we have

- $Ap_E = A\pi_D$ and
- $\rho(\pi_D)A = \sigma`(p_E)A$,

especially if

$$\rho(\pi_D)\sigma`(p_E) = \pi_D p_E \tag{6}$$

we have

$$\mathfrak{M} = \left\{ \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \mid e \in E \right\},$$

and if

$$\rho(\pi_D)\sigma`(p_E) = -\pi_D p_E \tag{7}$$

we have

$$\mathfrak{M} = \left\{ \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \mid e \in E \right\}.$$

We define now a signed hermitian form on W. We define f from $\mathrm{M}_2(E)$ to E to be the sum of all entries, and we define h_E to be $f \circ \tilde{h}$, more precisely the following two cases occur:

Case (6):

$$h_E = \mathrm{tr} \circ \tilde{h}|_{W \times W}.$$

Case (7):

$$h_E = a_{21} \circ \tilde{h}|_{W \times W},$$

where a_{21} is the map from $\mathrm{M}_2(E)$ to E which maps a matrix to its entry on position $(2,1)$.

Step 2: Let Λ be as assumed in the proposition. Then by [BL02] there is an o_E-lattice function Γ such that Λ equals $\Gamma \otimes_{o_k} o_L$. We want to apply Lemma 1.7, and we need to prove is the self-duality of Γ with respect to h_E, i.e. we have to prove

$$h_E(w, M) \subseteq \mathfrak{p}_E \text{ if and only if } h(w, M \otimes_{o_k} o_L) \subseteq \mathfrak{p}_D,$$

for all $w \in W$ and all full o_E-lattices M of W. It follows from

$$\mathfrak{M} \cap f^{-1}(\mathfrak{p}_E) = \mathfrak{M} \cap \bigcap_{u \in o_E^\times} u^{-1}(\lambda \otimes_k \mathrm{id}_D)^{-1}(\mathfrak{p}_D).$$

To prove the last equation we use the decomposition

$$\mathrm{M}_2(E) = E \oplus El' \oplus E\pi_D \oplus El'\pi_D.$$

For example for case (6): Let A be an element of $\mathfrak{M}$ with coefficients e_1, e_2, e_3, e_4 in the above decomposition, then we have

$$e_2 = e_4 = 0, \ e_1 - e_3 p_E = 0,$$

and every such matrix is an element of $\mathfrak{M}$. Thus have $f(A) \in \mathfrak{p}_E$ if and only if $e_1 \in \mathfrak{p}_E$, i.e. if and only if

$$\lambda(e_1 o_E) \subseteq \mathfrak{p}_k \text{ and } \lambda(e_3 o_E) \subseteq o_k,$$

i.e. if and only if

$$\lambda \otimes_k \mathrm{id}_D(o_E A) \subseteq \mathfrak{p}_D.$$

q.e.d.

Case 2.2: Here we follow a similar strategy as in case 2.1.. We fix an unramified two-dimensional field-extension L of k in D which is ρ-invariant and a prime element π_D of D which normalizes L, such that the square of π_D is a prime element of k denoted by π_k and $\rho(\pi_D)$ equals $+\pi_D$ or $-\pi_D$. We can embed L into E, because $2|f(E|k)$,

and we identify L with its image under the embedding. As in case 2.1. we identify $E \otimes_k D$ with $\mathrm{M}_2(E)$ but now via

$$\pi_D \mapsto \begin{pmatrix} 0 & \pi_k \\ 1 & 0 \end{pmatrix}, l' \mapsto \begin{pmatrix} l' & 0 \\ 0 & -l' \end{pmatrix}.$$

We now prove Lemma 1.11 for this case.

Proof: [Case 2.2] Step 1: We consider the idempotents 1^1 and 1^2 in $E \otimes_k L$, i.e. 1^1 is the matrix $E_{1,1}$ and 1^1 is the matrix $E_{2,2}$ in the standard notation of linear algebra. We define $V^i := 1^i V$ and $W := V^1$, and we have to recall that $V^2 = V^1 \pi_D$. Let $\mathfrak{M}$ be the image of $W \times W$ under $\tilde{h}$. The equations

- $A1^1 = A$ and
- $(\sigma' \otimes_k \rho)(1^1)A = A$,

for $A \in \mathfrak{M}$ imply that

$$\mathfrak{M} = \left\{ \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \mid e \in E \right\}, \tag{8}$$

or

$$\mathfrak{M} = \left\{ \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \mid e \in E \right\}. \tag{9}$$

We take the same f as in Case 2.1. to define h_E.

Step 2: Under the decomposition $V = V^1 \oplus V^2$ the lattice function Λ has the form

$$\Lambda(t) = \Theta(t) \oplus \Theta(t - \frac{1}{2})\pi_D$$

by [BL02]. Here Θ is an element of $\mathrm{Latt}^1_{o_E}(E)$. The only property left to prove is that Θ is self-dual.

Step 2a: At first we show the equivalence of the following two statements for an element A of $\mathfrak{M}$.

1. $f(A) \in \mathfrak{p}_E$.
2. $\tilde{\lambda}(o_E A \pi_D^{-1}) \in \mathfrak{p}_D$.

For this we look at the decomposition of $E \otimes_k D$ as in case 2.1. and we obtain for A the relations

(8) $e_3 = e_4 = 0$ and $e_1 = l' e_2$.

(9) $e_1 = e_2 = 0$ and $e_3 = l' e_4$.

From these relations the equivalences follow from (1).

Step 2b: We have to show

$$\Theta^{\#}(t) = \Lambda^{\#}(t) \cap V^1.$$

$\supseteq$: This follows directly from 2.$\Rightarrow$1..

$\subseteq$: Let w be an element of $\Theta^{\#}(t)$, i.e. we have

$$h(w, \Theta((-t)+)\pi_D^{-1}) \subseteq \mathfrak{p}_D,$$

i.e.

$$h(w, 1^1\Lambda((-t)+)) \subseteq \mathfrak{p}_D^2$$

and

$$h(w, 1^2\Lambda(-\frac{1}{2} + (-t)+)) \subseteq \mathfrak{p}_D,$$

especially

$$h(w, 1^2\Lambda((-t)+)) \subseteq \mathfrak{p}_D.$$

Thus

$$h(w, \Lambda((-t)+)) \subseteq \mathfrak{p}_D,$$

and therefore w is an element of $\Lambda^{\#}(t)$.

Step 3: Let us now consider Λ and Λ' from the assumption of the Lemma. Let Θ' be the intersection of Λ' with V^2. Since both pairs (E, Λ) and (E, Λ') have the same embedding type both lattice functions have the same invariants we conclude that there is an element $\tilde{g}$ of the centralizer $\tilde{H}$ of E in $\tilde{G}$ such that Λ equals $\tilde{g}\Lambda'$. The Broussous-Lemaire map j_E [BL02] from $\mathfrak{B}(\tilde{G})^{E^\times}$ to $\mathfrak{B}(\mathrm{Z}_{\tilde{H}})$ sends Λ to Θ and Λ' to Θ'. Thus Θ and Θ' are in the same $\tilde{H}$-orbit, since j_E is $\tilde{H}$-equivariant. Lemma 1.7 implies now that there is an element h of the centralizer of E in G such that $h\Theta$ equals Θ'. Now the element g of G defined by

$$g(w + w'\pi_D) := h(w) + h(w')\pi_D, \ w, w' \in V^1,$$

is our desired element sending Λ to Λ'. q.e.d.

Proof: [of 1.5 for Case 2] We copy the proof for the case $D = k$, but for lattice functions, and we apply Lemma 1.11 instead of Lemma 1.7. q.e.d.

References

[BG00] P. Broussous and M. Grabitz. Pure elements and intertwining classes of simple strata in local central simple algebras. *Comm. Algebra*, 28(11):5405–5442, 2000.

[BH96] C.J. Bushnell and G. Henniart. Local tame lifting for GL(N). I. Simple characters. *Inst. Hautes Études Sci. Publ. Math.*, (83):105–233, 1996.

[BL02] P. Broussous and B. Lemaire. Building of GL(m, D) and centralizers. *Transform. Groups*, 7(1):15–50, 2002.

[BS09] P. Broussous and S. Stevens. Buildings of classical groups and centralizers of Lie algebra elements. *J. Lie Theory*, 19(1):55–78, 2009.

[BT84] F. Bruhat and J. Tits. Schémas en groupes et immeubles des groupes classiques sur un corps local. *Bull. Soc. Math. France*, 112(2):259–301, 1984.

[BT87] F. Bruhat and J. Tits. Schémas en groupes et immeubles des groupes classiques sur un corps local. II. Groupes unitaires. *Bull. Soc. Math. France*, 115(2):141–195, 1987.

[Sch85] W. Scharlau. *Quadratic and Hermitian Forms.* Springer-Verlag, Berlin and Heidelberg, 1985.

[Ste08] Shaun Stevens. The supercuspidal representations of p-adic classical groups. *Inventiones Mathematicae*, 172:289–352, 2008.